Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(01)00038-7

J. Appl. Maths Mechs, Vol. 65, No. 2, pp. 339–351 2001

0021-8928/01/$—see front matter

THE STEADY SUBSONIC MOTION OF CRACKS AND NARROW NOTCHES ALONG THE INTERFACE IN A COMBINED ANISOTROPIC PLANE†

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(*Received* 16 *September* 1999)

Solutions of mixed dynamic problems are constructed in the case of a piecewise-homogeneous plane coinciding with the plane of symmetry of the elastic properties of the medium, that is, the most common case of anisotropy when the plane and the antiplane problems separate. Conditions for the stress and displacements vectors are specified in three combinations and of certain equivalent versions on the straight interface between the two materials. For example, the tractions on the open intervals are given, at the slippage, the shear resistance and the discontinuity in the normal displacement are specified and, in domains of cohesion, the discontinuities in the stresses and displacements are specified. By introducing new complex potentials, the common representations of solutions in steady-state dynamics [1, 2] can be successfully transformed to a type of boundary-value problems which has previously been studied [3–5]. The following problems are investigated as examples: the problem of a notch with an unknown section of contact of the edges due to an external stress field, and the problem of the motion of a semi-infinite cut under the action of forces applied to the cut surfaces, with alternative conditions of their contact (opening or slipping). © 2001 Elsevier Science Ltd. All rights reserved.

Problems in the steady-state dynamics of an isotropic, elastic half-plane and the statics of an anisotropic elastic half-plane with two types of boundary conditions were investigated for the first time by Galin [6] and Savin [7]. The first solution was constructed [8] for a combined plane consisting of two different isotropic materials with three types of contact conditions but with a single slip interval. Subsequently, a general solution of the Riemann–Hilbert vector problem was given [3] which can be factorized by methods of analytic continuation and conformal mapping of a domain. Similar problems with many types and combinations of conditions in a number of sections have subsequently been studied in the case of a half-plane or an orthotropic plane with narrow notches [5]. It has been shown that these problems reduce to a new combined boundary-value problem in the theory of functions of a complex variable, with conditions of the Riemann and Dirichlet type in different intervals, which admits of a general solution in terms of Cauchy integrals [4].

In this paper, we give an explicit definition of this solution with simplifications applying to the aims of the investigation and, also, a derivation of the formulae for the boundary values of the required quantities and the asymptotic forms at the nodal points.

Three types of conditions are also considered in the treatment of the static problem of a crack at the interface of a combined anisotropic plane using the method of boundary integral equations [9].

1. FORMULATION OF THE PROBLEM

We will consider the plane problem of the interaction of two anisotropic, linearly elastic half-planes made of different materials. In the systems of intervals $\mathbf{S} = \cup S_k$, $S_k = [s_k, t_k]$, $k = 1, \dots, K$, $s_1 < t_1 < \dots < t_K$, the half-planes are bonded together and, here, the discontinuities in the stress and displacement vectors are specified while, in the sections under the slip condition $\mathbf{L} = \cup L_j$, $L_j = \langle a_j, b_j \rangle$, $j = 1, \dots, J$, the shear stresses and the discontinuities in the normal displacements and stresses are specified. In the remaining intervals, $\mathbf{T} = \cup T_m$, $m = 1, \dots, M$, the surfaces are open and the tractions on them are given; x, y is a rectangular system of coordinates moving at a constant, sub-Rayleigh velocity, c_0, and materials 1 and 2 occupy the half-planes $y > 0$ and $y < 0$.

The formalism in [1, 2] is based on the following notation for the Navier equations in the case of the steady-state spatial dynamics of an arbitrary anisotropic body

$$C^{\circ}_{ijkl} u_{k,il} = 0, \quad C^{\circ}_{ijkl} = C_{ijkl} - \rho c_0^2 \delta_{jk} \delta_{1i} \delta_{1l}, \quad i, j, k, l = 1, 2, 3 \tag{1.1}$$

†*Prikl. Mat. Mekh.* Vol. 65, No. 2, pp. 346–359, 2001.

where $\{C_{ijkl}\}$ is the stiffness tensor, $\mathbf{u} = (u_1, u_2, u_3)$ is the displacement vector, ρ is the density and δ_{jk} is the Kronecker delta. The effective tensor $\{C^\circ_{ijkl}\}$ loses some of the symmetry properties and remains invariant only with respect to change of subscripts $C^\circ_{ijkl} = C^\circ_{lkji}$. It is well known [1, 2] that the general solution of Eqs (1.1), which only depends on two variables, can be expressed in terms of three holomorphic functions $\phi_k(z_k)$, $z_k = x + p_k y$

$$\mathbf{t} = (\sigma_{yx}, \sigma_{yy}, \sigma_{y3}) = 2\mathcal{R}\{\mathbf{G}\boldsymbol{\phi}\}, \quad \mathbf{u}_{,x} = 2\mathcal{R}\{\mathbf{A}\boldsymbol{\phi}\}, \quad \boldsymbol{\phi} = \{\phi_k(z_k)\} \tag{1.2}$$
$$\mathbf{s} = (\sigma_{xx}, \sigma_{xy}, \sigma_{x3}) = 2\mathcal{R}\{\rho c_0^2 \mathbf{A}\boldsymbol{\phi} - \mathbf{G}\boldsymbol{\Gamma}\boldsymbol{\phi}\}$$

where t and $\mathbf{s}$ are stress vectors, and the matrices $\mathbf{A}$, $\mathbf{G}$ and Γ depend on the solution of the eigenvalue problem

$$\sum_{q=1}^{3} \{C^\circ_{1jk1} + p_q(C^\circ_{1jk2} + C^\circ_{2jk1}) + p_q^2 C^\circ_{2jk2}\} A_{kq} = 0 \tag{1.3}$$
$$\det\{C^\circ_{1jk1} + p_q(C^\circ_{1jk2} + C^\circ_{2jk1}) + p_q^2 C^\circ_{2jk2}\} = 0$$

The normalization of each column of matrix $\mathbf{A}$ is arbitrary. When $c_0 < c_R$, where c_R is the smaller of the velocities of the Rayleigh waves in the direction of the x axis, all the eigenvalues, generally speaking, are complex and, among the three pairs of conjugate roots of the second equation of (1.3), the three roots with the positive imaginary part p_k ($k = 1, 2, 3$) are selected. The coefficients of the matrix $\mathbf{G}$ are then determined using the formulae (for the definition of Γ, see [2])

$$G_{jq} = \sum_{k=1}^{3} \{C^\circ_{2jk1} + p_q C^\circ_{2jk2}\} A_{kq} = -\sum_{k=1}^{3} \{p_q^{-1} C^\circ_{1jk1} + C^\circ_{1jk2}\} A_{kq} \tag{1.4}$$

If x, y is the plane of symmetry of the elastic properties, then $C^\circ_{ijkl} = 0$ for the sets of subscripts containing an odd number of numerals 3. The equation for u_3 in (1.1), and this means also the equation for the antiplane problem, are separated from the equations for u_1, $u_2 \equiv u$, v (the plane problem) and the determinant in (1.3) is decomposed into two cofactors such that the root p_3 and the sole component A_{33} of the corresponding eigenvector are calculated independently.

We will henceforth confine ourselves to the treatment of the plane problem and seek the vectors $\mathbf{t} = (\sigma_{xy}, \sigma_{yy}) \equiv (\tau, \sigma)$, $\mathbf{U} = (u_{,x}, v_{,x}) \equiv (U, V)$, starting from the representations in terms of the complex potentials, by putting $i, j, k, l, q = 1, 2$ in (1.2)–(1.4). The unusual cases of multiple and real roots p_1, p_2, which require a different approach, are not considered. Then, the solutions which are constructed below do not degenerate when $c_0 \to 0$, as in the case of isotropy, and include the correspondent static problems.

The boundary conditions of the main problem

$$\mathbf{t}^\pm = \mathbf{t}_0^\pm(x) \ \ (\mathbf{T}), \quad \tau^\pm = \tau_0^\pm(x), \quad [V] = V_*(x) \ \ (\mathbf{L}) \tag{1.5}$$
$$[\mathbf{t}] = \mathbf{t}_*(x), \quad [\mathbf{U}] = \mathbf{U}_*(x) \ \ (\mathbf{S}), \quad \mathbf{t} \to \mathbf{t}_\infty, \quad x^2 + y^2 \to \infty$$

where the plus and minus superscripts denote contraction from above and below onto the axis $y = 0$ and the right-hand sides satisfy the Holder conditions, are then split into parts which are symmetric and antisymmetric with respect to the stresses, using the notation

$$f_s^\pm = \langle f \rangle \equiv \frac{1}{2}\{f^+(x,0) + f^-(x,0)\}, \quad \pm 2 f_{as}^\pm = [f] \equiv f^+(x,0) - f^-(x,0)$$

Correspondingly, we construct the fields $\mathbf{t}_s$, $\mathbf{U}_s$ and $\mathbf{t}_{as}$, $\mathbf{U}_{as}$, the sum of which is the solution of the initial problem (1.5). In the case of a problem with asymmetric stresses and null conditions for $[\mathbf{U}_{as}]$ at the interface with the condition that the solution vanishes at infinity, it follows from (1.2) that there is a separate Dirichlet problem for the upper and lower half-planes, the solution of which is given by integrals of the Cauchy type [10]. The right-hand sides of the boundary conditions in the case of a problem with symmetric conditions for the stresses (the subscript s is henceforth omitted) then become known

$$\mathbf{t}^\pm = \langle \mathbf{t}_0 \rangle (\mathbf{T}), \quad [t] = 0, \quad [\mathbf{U}] = \mathbf{U}_* \ \ (\mathbf{S})$$
$$\tau^\pm = \langle \tau_0 \rangle, \quad [\sigma] = 0, \quad [V] = V_* \ \ (\mathbf{L}) \tag{1.6}$$

For closure, it is necessary to supplement this problem with a set of specified discontinuities [**u**] at each joint of the intervals **S** and **T** and of the discontinuities [v] at the remaining nodal points.

We now introduce the vector functions $\mathbf{h} = (h_1, h_2)$ and $\chi(z) = (\chi_1, \chi_2)(z = x + iy)$ and also the matrices **B** and **H** using the formulae

$$\begin{aligned}
&\mathbf{h} = 2\mathbf{G}\boldsymbol{\phi}, \quad \chi_1 = ih_1, \quad \chi_2 = h_2 + \beta_0 h_1 \\
&\mathbf{B} = i\mathbf{A}\mathbf{G}^{-1}, \quad \mathbf{H} = \mathbf{B}_1 + \overline{\mathbf{B}}_2 = \{H_{kj}\}, \quad H_{jj} \equiv H_j > 0 \\
&H_{12} = \overline{H}_{21} \equiv \hat{H} + i\tilde{H}, \quad \det H > 0, \quad \beta_0 = \hat{H} / H_2
\end{aligned} \tag{1.7}$$

Here and henceforth, for brevity the notation $g = \hat{g} + i\tilde{g}$ is introduced for any complex quantity. The matrix **H**, which is defined above in terms of the matrices **B**, referring to materials 1 and 2, is Hermitian and positive definite. We also note that H_1, H_2, $\hat{H}$ are invariant under a permutation of the positions of materials 1 and 2 while $\tilde{H}$ changes sign. Below, without loss of generality, we put $\tilde{H} > 0$.

The analytical continuation $\mathbf{h} = \overline{\mathbf{h}(\bar{z})} \equiv \bar{\mathbf{h}}$ follows from the symmetry of the boundary conditions in the stresses (1.6) and (1.2), (1.7). The stresses and the discontinuities in the derivatives of the displacements on the boundary can be expressed in terms of the boundary values of the function **h** and, then, using formulae (1.7) and relations (1.8) presented below, the potentials χ_k can be expressed as follows:

$$\mathbf{t} = \mathcal{R}\{\mathbf{h}^+\}, \quad [\mathbf{U}] = \mathcal{T}\{\mathbf{H}\mathbf{h}^+\} \tag{1.8}$$

$$\begin{aligned}
&\sigma(x) = \hat{\chi}_2^+(x) - \beta_0\tilde{\chi}_1^+(x), \quad \tau(x) = \tilde{\chi}_1^+(x) \\
&[V(x)] = H_2\{\beta\tilde{\chi}_1^+(x) + \tilde{\chi}_2^+(x)\}, \quad [U(x)] = \tilde{H}\{\zeta\hat{\chi}_1^+(x) + \hat{\chi}_2^+(x)\} + \beta_0[V(x)] \\
&\beta = -\frac{\tilde{H}}{H_2}, \quad \zeta = \frac{\hat{H}^2 - H_1H_2}{\tilde{H}H_2}, \quad \zeta < \beta < 0
\end{aligned} \tag{1.9}$$

We now use the fact that the boundary conditions when $y = \pm 0$ in the planes z and z_k are written identically. Actually, the problem will be considered in the z plane and the functions $\chi_j(z)$ will be found. The functions $h_j(z)$ are then found from (1.7) and the initial potentials $\phi(z) = \mathbf{G}^{-1}\mathbf{h}(z)$ are recovered. Finally, the variable z is replaced by z_k and the field outside the boundary can be calculated using formulae (1.2).

The generalized Riemann–Hilbert boundary-value problem follows from relations (1.6)–(1.9): it is required to find the vector $\chi(z)$, which is holomorphic in the upper half plane z using the following conditions on the real axis

$$\begin{aligned}
&\mathcal{T}\{\mathbf{D}\chi^+(x)\} = \mathbf{f}(x), \quad -\infty < x < +\infty \\
&\mathbf{D} = \mathbf{D}_1 = \begin{Vmatrix} 1 & 0 \\ 0 & i \end{Vmatrix}, \quad \mathbf{f} = \begin{Vmatrix} f_1 \\ f_2 \end{Vmatrix} (\mathbf{T}), \quad \mathbf{D} = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, \quad \mathbf{f} = \begin{Vmatrix} f_1 \\ f_3 \end{Vmatrix} (\mathbf{L}) \\
&\mathbf{D} = \mathbf{D}_2 = \begin{Vmatrix} \beta & 1 \\ i\zeta & i \end{Vmatrix}, \quad \mathbf{f} = \begin{Vmatrix} f_4 \\ f_5 \end{Vmatrix} (\mathbf{S}), \quad f_3 = H_2^{-1}V_* - \beta f_1 \\
&f_1 = \langle\tau_0\rangle, \quad f_2 = \langle\sigma_0\rangle + \beta_0 f_1 \\
&f_4 = H_2^{-1}V_*, \quad f_5 = \tilde{H}^{-1}\{U_* - \beta_0 V_*\}
\end{aligned} \tag{1.10}$$

and satisfies the constraints on its behaviour at the nodal points and at infinity, which are usual in the theory of elasticity. Unlike the case of the similar scalar problem, no general method is known for solving the generalized coupled Riemann–Hilbert vector problem with three or more types of boundary conditions. It is important, however, that the initial problem can be reduced to problem (1.10), which belongs to a special class of such problems: the upper rows of the piecewise-constant matrix **D** are real numbers and the lower rows are imaginary numbers. This is achieved by introducing the new potentials χ_k (the functions h_k and the matrices **B** and **H** were introduced earlier [1, 2]). Factorization of the vector problem then becomes possible [3, 4].

The solution of problem (1.10) in Section 4 will be constructed for the case of a finite length of all the intervals from **L** and **S**. Other cases, including discontinuities of the boundary conditions at infinity, can be treated using a corresponding passage to the limit or an elementary conformal mapping of the plane, converting an infinite point into a finite point. We also note that the specification of the far stress field is not always arbitrary and must be matched with the boundary conditions.

By substitution $\tau \leftrightarrow \sigma$, $U \leftrightarrow V$ (the subscripts on H_k, h_k, χ_k change places) and, also, using the duality $\tau \leftrightarrow U$, $\sigma \leftrightarrow V$ and, then, the substiution $\tau \leftrightarrow \sigma$, $U \leftrightarrow V$ and inversion of the matrix **B**, it is possible to obtain formulations of the new boundary conditions which are equivalent to (1.10). We shall initially consider the case when the solution is constructed by the method of conformal mapping, which is simpler and more readily understandable.

2. ONE SLIP SECTION

suppose there is a slip section [–1, 1]. First we shall remove the inhomogeneity in the boundary condition by considering the difference

$$\mathbf{X} = \chi - \chi^\circ, \quad \chi^\circ = (\chi_1^\circ, \chi_2^\circ) = \frac{1}{\pi}\int_{-1}^{1} \frac{\mathbf{f}(t)dt}{t-z}$$

Boundary-value problem (1.10) with the modified right-hand sides ($\mathbf{f} \to \mathbf{f}_0$)

$$\mathbf{f}_0 = (f_1, f_2 - \chi_2^\circ) \ (\mathbf{T}), \quad \mathbf{f}_0 = 0 \ (\mathbf{L}), \quad \mathbf{f}_0 = (f_4, f_5 - \zeta\chi_1^\circ - \chi_2^\circ) \ (\mathbf{S}) \tag{2.1}$$

is obtained for the vector function $\mathbf{X}(z)$. A method of solving this problem was proposed earlier [3]. Briefly, it is as follows. We continue the function $\mathbf{X}(z)$ analytically across the interval [–1, 1]: $\mathbf{X}(z) = \overline{\mathbf{X}(\bar{z})}$. The Zhukovskii conformal mapping: $z = (\omega + \omega^{-1})/2$ ($\omega = \xi + i\eta = z + \sqrt{z^2-1}$) converts the z plane with the cuts $|x| > 1, y = \pm 0$ into the upper half-plane ω, and the interval [–1, 1] into an upper semicircle of unit radius with correspondence of the points $x = s_k^\pm, t_k^\pm \Leftrightarrow \xi = (s_k')^{\pm 1}, (t_k')^{\pm 1}$. In the case of a vector $\mathbf{\Phi} = \mathbf{D}_1\mathbf{X}$, where $\mathbf{X}(\omega) = \overline{\mathbf{X}(1/\bar\omega)}$, $\eta \geqslant 0$ and $\Phi(\omega) = \overline{\phi(1/\bar\omega)}$, which is piecewise-holomorphic in the ω plane, we obtain the matching problem in which the coupling matrix on the boundary is equal to the identity matrix in the sections from **T** and is equal to a certain matrix $\mathbf{D}^\circ$ on $S\omega = (s_k', t_k') \cup (s_k'^{-1}, t_k'^{-1})$. By diagonalizing the matrix $\mathbf{D}^\circ$, the problem reduces to determining just one scalar function $W(\omega)$ using the matching boundary conditions, the solution of which is constructed in the standard way [11] and is presented below

$$\mathbf{X} = \begin{Vmatrix} W + \overline{W} \\ \varkappa W - \varkappa \overline{W} \end{Vmatrix}, \quad W = \Pi\{R + I\}, \quad \Pi = \Pi_1\Pi_2 \tag{2.2}$$

$$\Pi_1 = \prod_1^K \left[\frac{(\omega - s_k')(t_k'\omega - 1)}{(s_k'\omega - 1)(\omega - t_k')}\right]^{i\alpha}, \quad 2^{K+1}\Pi_2 = \left\{(z^2-1)\prod_1^K (z - s_k)(z - t_k)\right\}^{-1/2}$$

$$R = \sum_{-K-1}^{K+1} r_k\omega^k, \quad r_k = -\bar{r}_{-k}, \quad I = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{W^\circ(t)dt}{\Pi^+(t)(t-\omega)}$$

$$\varkappa = \sqrt{\zeta\beta}, \quad \mathbf{W}^\circ = \mathbf{T}_1^{-1}\mathbf{B}_0\mathbf{g} = (W^\circ, W_*^\circ) \quad (\mathbf{S}_\omega), \quad \mathbf{W}^\circ = \mathbf{T}_1^{-1}\mathbf{g} \quad (\mathbf{T}_\omega)$$

$$\mathbf{B}_0 = \mathbf{D}_1\mathbf{D}_2^{-1}, \quad \mathbf{g} = \mathbf{f}_0[x(\xi)] = (g_1, g_2), \ |\xi| > 1$$

$$\mathbf{T}_1 = \begin{Vmatrix} 1 & 1 \\ \varkappa & -\varkappa \end{Vmatrix}, \quad \alpha = \frac{\ln|\lambda|}{2\pi}, \quad \lambda = \frac{d-1}{d+1} < 0, \quad d = \frac{\varkappa}{\zeta}, \quad |\lambda| > 1$$

$$g_m(\xi) = (-1)^m g_m(1/\xi), \ |\xi| < 1$$

where d, an analogue of Dundurs' parameter, determines the degree of difference in the elastic properties of materials 1 and 2, and, in order for the integral to exist, a constraint is required on the nature of the decay of the specified functions as $x \to \pm\infty$. We construct the cuts for isolating the single-value branch of the function $\Pi_1(\omega)$ along S_ω and the condition $1^\gamma = 1$ fixes the set of branches of its cofactors. The free $2K+3$ real constants in solution (2.2) are determined from the two conditions for matching the stresses at infinity with the specified remote field $\mathbf{t}_\infty$ and the $2K+1$ conditions for the single-valuedness of the displacements on going around the open and closed intervals [10], taking account of the discontinuities in the displacements specified at the nodal points.

If the two points $x = \pm 1$ are boundary points for the intervals **L** and **S**, then the corresponding factors $(z - s_l)$ and $(z - t_m)$, $s_l = -1$, $t_m = 1$ have to be removed from the product $\Pi_2(z)$. When the slip zone **L** is also contiguous with a cohesive part from **S** and with an open zone from **T**, then, apart from the removal of the corresponding factor as in the preceding case, the factor $\sqrt{z+1} - \sqrt{z-1}$ is added to the function $\Pi_2(z)$, which compensates for the irregular behaviour of the solution at infinity. The polynominal (either the upper or the lower signs are chosen)

$$R = \sum_{-K-1}^{K} r_k\omega^k,\quad r_{k-1} = \pm\bar{r}_{-k},\quad k = 0,\ldots,K+1,\quad x = \pm 1 \in \mathbf{L}/\mathbf{S},\quad x = \mp 1 \in \mathbf{T}/\mathbf{L}$$

is also subject to a change and the number of free constants is equal to $2K+2$ if the boundary conditions do not lose the discontinuity at infinity and are identical with the number of additional conditions of the problem.

Close to the nodal points $z = \pm 1$, we have an ordinary root singularity of the solution: at the boundary points of **S/L** (the interval from **S** is located to the left and the interval from **L** to the right) and **L/S**, the oscillating singularity automatically vanishes since the same cofactors are cancelled in the case of the function $\Pi_1(\omega)$. This singularity also disappears at points of the type of **T/S** on the surface $d = 0 \Leftrightarrow |\lambda| = 1$, $\alpha = 0$ in the phase space of the elasticity constants.

3. A NARROW NOTCH AT THE INTERFACE

We will now consider an example of a static problem with a single frictionless slip zone $[-1, 1]$, which is formed by the closure of the sides of the notch (a, b), $a < -1$, $b > 1$ under the action of a remote field τ_∞, $\sigma_\infty < 0$. Initially, it had a constant opening δ_0 which it subsequently only retains at its ends. Solution (2.2) takes the form

$$\Pi_1 = \varphi^{i\alpha},\quad \varphi = \frac{(A-\omega)(1-B\omega)}{(A\omega-1)(\omega-B)},\quad \Pi_2 = \frac{1}{4\sqrt{(z^2-1)(a-z)(z-b)}} \tag{3.1}$$

$$R = r_2\omega^2 - \bar{r}_2\omega^{-2} + r_1\omega - \bar{r}_1\omega^{-1} + ir_0,\quad l = 0$$

$$\bar{r}_0 = 0,\ A = a + \sqrt{a^2-1},\quad B = b - \sqrt{b^2-1}$$

From the conditions at infinity, which include the fact that the coefficient of z^{-1} in the expansion of $W(z)$ when $z \to \infty$ is equal to zero, we extract the complex coefficients r_1 and r_2:

$$r_2 = \frac{\sigma_\infty + \beta_0\tau_\infty + i\varkappa\tau_\infty}{i\varkappa(|\lambda|^{1/2} + |\lambda|^{-1/2})}\exp\left\{i\alpha\ln\left|\frac{A}{B}\right|\right\} \tag{3.2}$$

$$r_1 = r_2\{2i\alpha(\sqrt{a^2-1} + \sqrt{b^2-1}) - a - b\}$$

The conditions that the displacements should be single-valued on passing around the whole of the contour of the notch will be satisfied if the term $O(z^{-1})$ is missing in the expansion of the potentials [10] and the condition for the discontinuity in the normal displacements to grow by an amount δ_0 in the interval $(1, a)$ remains for calculating the coefficient r_0

$$\int_1^a [V]dx = \delta_0,\quad [V(x)] = 2\varkappa H_2\tilde{W}^+(x) \tag{3.3}$$

$$r_0 = \{(\varkappa H_2)^{-1}\delta_0 - I_1\}I_2^{-1},\quad I_k = 2\int_1^a \Pi_2^+(x)\psi_k(x)dx$$

$$\psi_1 = \mathcal{T}\{\Pi_1^+(x)(R[\xi(x)] - ir_0)\exp(i\alpha\ln|\varphi(x)|)\},\quad \psi_2 = \cos(\alpha\ln|\varphi(x)|)$$

In the calculations, it is useful to take account of the smallness of the constant α in statics and low-speed dynamics and to neglect terms of the order of α^2 compared with unity in (3.1)–(3.3). However, as the velocity c_0 approaches c_R, the magnitude of α increases to infinity [1, 2].

We determine the unknowns a and b from the condition for there to be no singularity at the boundary points of incomplete contact of the smooth surfaces $z = \pm 1$: $2\tilde{r}_2 \pm 2\tilde{r}_1 + r_0 = 0$. In the physical plane where the ends of the notch a° and b° are specified numbers and $x^\circ_{1,2}$ are the required boundaries of the domain of contact of the notch surfaces, we proceed with the transformation

$$z^\circ = \frac{(a^\circ - b^\circ)z + ab^\circ - a^\circ b}{a-b}; \quad x = \pm 1 \Leftrightarrow x^\circ_{1,2} = \frac{ab^\circ - a^\circ b \pm (a^\circ - b^\circ)}{a-b}$$

A solution obviously exists if the numbers $x^\circ_{1,2}$, obtained satisfy the inequalities $a^\circ < x^\circ_1 \leq x^\circ_2 < b^\circ$. The beginning of the closing $x^\circ_1 = x^\circ_2$ determines the limiting surface $\sigma^*_\infty = \sigma^*_\infty(\tau_\infty, \ldots)$ in the parameter space.

In the case of symmetry ($\tau_\infty = 0, a = -b$)

$$r_2 = i\tilde{r}_2 = \frac{-i\sigma_\infty |\lambda|^{1/2}}{\varkappa(1-\lambda)}, \quad r_1 = \hat{r}_1 = -4\alpha\sqrt{a^2-1}\,\tilde{r}_2, \quad r_0 = \frac{1}{I_2}\left(\frac{\delta_0}{\varkappa H_2} - I_1\right)$$

$$I_1 = 2\tilde{r}_2\{a\mathbf{E}(\sqrt{1-a^{-2}}) - I_2\}, \quad I_2 = (2a)^{-1}\mathbf{K}(\sqrt{1-a^{-2}})$$

where $\mathbf{K}$ and $\mathbf{E}$ are complete elliptic integrals of the first and second kind.

We use the equation for determining the point of separation $2\tilde{r}_1 + r_0 = 0$ to solve the inverse problem of determining the magnitude of the stress σ_∞ as a function of the length of the overlap of the notch $2l$.

$$-\sigma_\infty \approx \frac{\delta_0(|\lambda|^{1/2} + |\lambda|^{-1/2})}{2H_2 I_0(l)}, \quad I_0 = \int_l^1 \sqrt{\frac{t^2 - l^2}{1-t^2}}\,dt \tag{3.4}$$

where, for convenience, the half-length of the notch has been taken as the unit of length (renormalization), the opening δ_0 has already been divided by this half-length, and the accuracy of the formula is $O(\alpha_2)$. When $l \to 0$, the limit $I_0 \to 1$ exists and it is then simple to find the limit stress σ^*_∞ from (3.4). It is identical with the value which can be extracted from the solution of the corresponding problem of a notch where its sides do not come into contact. On the other hand, if $l \to 1$, then $I_0 \sim \frac{1}{2}\pi(1-l)$ and $\sigma_\infty \sim (1-l)^{-1} \to \infty$.

4. THE GENERAL CASE

Suppose the number of intervals L_j is arbitrary, $J > 1$. From these intervals, we separate the semi-open parts (having just a single point in common with $\mathbf{T}$) $\langle a'_n, b'_n \rangle \in \mathbf{L}' \subset \mathbf{L}$ ($n = 1, \ldots, J'$), the closed parts (with two such points) $[a''_l, b'_l] \in \mathbf{L}'' \subset \mathbf{L}$ ($l = 1, \ldots, J''$) and the open parts $J - J' - J''$. Following the procedure described in [4], we express the solution of the vector problem (1.10) in terms of the single scalar function $F(z)$

$$\chi_j(z) = \varkappa^{j-1}\{F(z) - (-1)^j \overline{F(\bar{z})}\}, \quad j = 1, 2 \tag{4.1}$$

Substituting expressions (4.1) into conditions (1.10) we obtain a combined boundary-value problem for determining the piecewise-holomorphic function using the conditions for a Dirichlet problem in $\mathbf{L}$ and the conditions for a Riemann problem in $\mathbf{R} = \mathbf{T} \cup \mathbf{S}$

$$\tilde{F}^\pm(x) = f^\pm(x), \quad 2f^\pm = \varkappa^{-1} f_3 \pm f_1 \quad (\mathbf{L}) \tag{4.2}$$

$$F^+(x) = F^-(x) + f_T(x), \quad f_T = \varkappa^{-1} f_2 + i f_1 \quad (\mathbf{T}) \tag{4.3}$$

$$F^+(x) = \lambda F^-(x) + f_S(x), \quad f_S = \frac{d f_5 + i f_4}{\beta + \varkappa} \quad (\mathbf{S}) \tag{4.4}$$

Unlike the conventional boundary-value coupling problem, the general solution of the combined homogeneous problem corresponding to (4.2)–(4.4) is found using the two canonical solutions since, in this case, it is impossible to select the single canonical solution with the largest permissible singularities at all of the nodal points and, consequently, having the greatest order at infinity [4, 10]. These solutions are sought in the form of the product of the canonical solution of the Riemann problem (4.3), (4.4), $Z(z)$, the auxiliary functions $\Pi_1(z)$ and $\Pi_*(z)$, which ensure that the necessary conditions and the proper order at the nodal points are satisfied, and, finally, the canonical solution of the Dirichlet problem (4.2)

$$F_c(z) = Z(z)\Pi_1(z)\Pi_*(z)e^{i\psi(z)}, \quad Z = Z_0\Pi_0 \tag{4.5}$$

$$Z_0 = \prod_{k=1}^{K}\left(\frac{z-s_k}{z-t_k}\right)^{i\alpha}, \quad \Pi_0 = \prod_{k=1}^{K}\frac{(z-s_k)^{-1/2}}{(z-t_k)^{1/2}}, \quad \Pi_1 = \prod_{j=1}^{J-1}\frac{1}{z-c_j}$$

$$\hat{\psi}^{\pm} = \eta_j^{\pm} \equiv \pi m_j^{\pm} - \arg\{Z\Pi_1\Pi_*\}^{\pm} \quad (\mathbf{L}) \tag{4.6}$$

Here, the integers $m_j^{\pm}$ are as yet arbitrary. We order the auxiliary complex constants c_j in the lower semicircles with the ends a_j, b_j and the function $\Pi_*(z)$ is specifically defined below. Cuts are constructed along the x axis joining the points a and ∞ for selecting branches of the power functions of the form $(z-a)^{\beta}$; $\arg(x-a)^{\pm} = 0, 2\pi, x > a$; $\arg(x-a)^{\pm} = \pi, x < a$. The factor, corresponding to the points $x = +\infty$ or $x = -\infty$ in a semi-infinite interval, drops out of the products but is taken account of as the limit argument of this factor. For example, if $s_1 = -\infty$, the factor $(z-s_1)$ drops out and, by $\arg(z-t_1)^{1/2}$, we mean $\lim\arg\{(z-t_1)(z-s)\}^{1/2}, s\to\infty$. Furthermore, if both of the semi-infinite intervals belong to the same set, then, in numbering them, we shall assume that they are a single interval.

We now construct the two canonical solutions such that one of them (we retain the notation F_c) ensures the specified power behaviour with the exponent of $-1/2$ at all of the nodal points s_k, t_k, a_j, b_j with the exception of the right-hand ends of the semiclosed intervals $\langle a_n', b_n'\rangle$

$$F_c(z) = Z_0(z)\Pi(z)e^{i\psi(z)}, \quad \Pi = \Pi_0\Pi_1\Pi_2\Pi_3 \tag{4.7}$$

$$\Pi_2 = \prod_{j=1}^{J''}\frac{(z-b_j'')^{-1/2}}{(z-a_j'')^{1/2}}, \quad \Pi_3 = \prod_{n:\, b_n'\in \mathbf{L}/\mathbf{S}}\left(\frac{z-b_n'}{z-a_n'}\right)^{1/2}$$

We simplify the solution of the inhomogeneous Dirichlet problem (4.6)

$$\psi = \frac{1}{\pi i}\sum_{j=1}^{J}\int_{L_j}\left\{\frac{Y(z)\langle\eta_j(t)\rangle}{Y^+(t)} + \frac{[\eta_j(t)]}{2}\right\}\frac{dt}{t-z} \tag{4.8}$$

$$Y(z) = \prod_{j=1}^{J}\sqrt{(z-a_j)(z-b_j)}$$

by noting that the numbers $[\eta_j(t)]$ are integers; it is then possible to ensure the condition $[\eta_j(t)] = 0$ and to eliminate the second integral in (4.8) by the choice

$$[m_j] = \frac{1}{\pi}[\arg(\Pi_0\Pi_2\Pi_3)] = 2(K_- + J_- + e_0), \quad e_0(x) = \begin{cases} 0, & x\notin \mathbf{L}^{**} \\ 1/2, & x\in \mathbf{L}^{**}\end{cases} \tag{4.9}$$

where $K_{\pm}$ and $J_{\pm}$ are the numbers of whole intervals from the subsets $\mathbf{S}$ and $\mathbf{L}''$ respectively, arranged to the right of (+) or to the left of (–) the point x so that

$$K = K_+ + K_-, \quad x\notin \mathbf{S}$$

$$J = J_+ + J_-, \quad x\notin \mathbf{L}'', \quad J = J_+ + J_- + 1, \quad x\in\mathbf{L}''; \quad \mathbf{L}^{**} = \mathbf{L}''\cup\mathbf{L}^*$$

and $\mathbf{L}^*\subset\mathbf{L}'$ is the subset of the half-closed parts $[a_n', b_n']$, that is, such that $b_n'\in\mathbf{L}/\mathbf{S}$. We emphasize that the following values of the arguments of the functions are taken into account which have been calculated for the case when the lengths of all of the intervals from the systems $\mathbf{S}$ and $\mathbf{L}$ are finite

$$[\arg\{Z_0\,\Pi_1\}]=0,\quad \langle\arg\{Z_0\,\Pi_1\}\rangle=\arg\{Z_0\,\Pi_1\}$$

$$\frac{1}{\pi}\arg\Pi_0^{\pm}=\begin{cases}-K_+\ (+), & K_+-2K\ (-)\qquad x\notin\mathbf{S}\\ -K_+-½\ (+), & K_+-2K+½\ (-),\ x\in\mathbf{S}\end{cases}$$

$$\frac{1}{\pi}\arg\Pi_2^{\pm}=\begin{cases}-J_+\ (+), & J_+-2J''\ (-),\qquad x\notin\mathbf{L}''\\ -J_+-½\ (+), & J_+-2J''+½\ (-),\ x\in\mathbf{L}''\end{cases}$$

$$\frac{1}{\pi}\arg\Pi_3^{\pm}=\begin{cases}0, & x\notin\mathbf{L}^*\\ \pm½, & x\in\mathbf{L}^*\end{cases},\quad \langle\arg\{\Pi_0\Pi_2\}\rangle=-\pi(K+J'')$$

Taking account of relations (4.9), in the remaining integral we have

$$\langle\eta_j\rangle=\pi w_j-\arg\{Z_0\Pi_1\}(t)$$

$$w_j=m_j^+-\frac{1}{\pi}\arg\{\Pi_0\Pi_2\Pi_3\}=m_j^++K_++J''-J_--e_0$$

The requirement $\psi(z)=O(1), z\to\infty$ is ensured by the conditions

$$\sum_{j=1}^{J}\int_{L_j}\frac{\langle\eta_j(t)\rangle t^l dt}{Y^+(t)}=0,\quad l=0,\ldots,J-2 \tag{4.10}$$

whence the numbers w_j are found by inspection and then, taking account of (4.9), the numbers $m_j^{\mp}$ as well as the quantities γ_j, which are related to c_j by the equalities

$$2c_j=a_j+b_j+(b_j-a_j)\exp(-i\gamma_j),\quad 0<\gamma_j<\pi$$

Here, it is useful to take account of the fact that the right-hand side of the first equality of (4.9) is an even number and that the number w_j is an integer in the sections $L_j\notin\mathbf{L}^{**}$ and fractional and a multiple of ½ if $L_j\in\mathbf{L}^{**}$. As shown in the example in Section 5, the numbers w_j are determined with a certain arbitrariness which only affects the signs of the canonical solutions.

The other canonical solution is the product $Y_0(z)F_c$, where

$$Y_0(z)=Y(z)Y_1(z),\quad Y_1(z)=\prod_{j=1}^{J'}(z-b_n')^{-1} \tag{4.11}$$

It, on the other hand, acquired root singularities when $z\to b_n'$ but loses them at all the remaining ends of the intervals $\langle a_j, b_j\rangle$. At the points S/T and T/S, the above-mentioned solutions have the same singularities. When $z\to\infty$, it follows from (4.7) and (4.11) that

$$F_c\sim z^{-r},\quad Y_0(z)F_c\sim z^{-s},\quad r=K+J+J''-1,\quad s=K+J'+J''-1$$

and, then, the solution of problem (4.2)–(4.4), which is bounded at infinity (it was previously required that it should vanish [4]), takes the form

$$F(z)=F_c(z)\{P_r(z)+iQ_s(z)Y_0(z)+F_1(z)+F_2(z)\} \tag{4.12}$$

$$F_1(z)=\frac{1}{2\pi i}\int_{\mathbf{R}}\frac{f_R(t)dt}{F_c^+(t)(t-z)},\quad f_R(t)=\begin{cases}f_S(t), & t\in\mathbf{S}\\ f_T(t), & t\in\mathbf{T}\end{cases}$$

$$F_2(z)=\frac{Y_0(z)}{\pi}\int_{\mathbf{L}}\frac{\langle f_*(t)\rangle dt}{Y_0^+(t)(t-z)}+\frac{1}{2\pi}\int_{\mathbf{L}}\frac{[f_*(t)]dt}{t-z}\equiv iY_0(z)F_{21}(z)+F_{22}(z)$$

$$f_*^{\pm}(t)=f^{\pm}(t)\{F_c^{\pm}(t)\}^{-1}-\tilde{F}_1(t)$$

where $P_r(z)$, $Q_s(z)$ are polynomials of degree r and s with real coefficients, the number of which is equal to

$$r+s+2=2K+J+J'+2J''=K+M+2J$$

since the equality $M = K - J + J' + 2J''$ can be proved. To determine these coefficients, we have the two conditions at the infinity, the $2J - 2$ conditions for the elimination of the artificial poles of the solution at the points $z = c_j$ and the $K + M$ conditions for the uniqueness of the displacements on passing around the zones of closure and slippage, taking into account of the specification of the discontinuities in these quantities at the nodal points. With this, the construction of the general solution has been completed.

Proceeding to an analysis of the solution, we first note that the general solution (4.12) contains the product of functions with oscillating (physically improper) singularities at the nodal points S/L and L/S, which are generated by the canonical solution $Z(z)$ of the Riemann problem (4.3), (4.4). These oscillations are damped with a factor $\exp[i\psi(z)]$ in the general solution, where the function $\psi(z)$ is the solution of the Dirichlet problem (4.6). Moreover, the corresponding factors, which oscillate with opposite phase, are concealed in the Cauchy type integral (4.8) and are only explicitly separated out in the asymptotic forms, which are presented below, where these "improper" oscillations are completely eliminated in this way. However, it is known that the zones of such oscillations can grow when shear forces predominate or in dynamics. There is no such superposition of singularities in the solutions presented in sections 2 and 3 which are more convenient, for example, in the numerical implementation, and it is also desirable to eliminate it in the specific definitions of the solution (4.12) as was done in Section 5. Although the solutions presented in Sections 2 and 4 must be identical when there is a single slip section, the transition from one solution to the other in a general form is extremely non-trivial and a set of variants arises. It is therefore simpler to solve the problem anew.

In the case of the application of the vectors of the forces $(\pm\Gamma_m, \Sigma_m)$ at the points $x_m \in \mathbf{T}$, $y = \pm 0$ and when there are no other loads, the functions determining the particular solution of the problem take the form

$$F_1(z) = \frac{1}{2\pi i}\sum_{m=1}^{M}\frac{(\beta_0 + i\varkappa)\Gamma_m + \Sigma_m}{\varkappa F_c^+(x_m)(x_m - z)}, \quad F_2(z) = \frac{Y_0(z)}{\pi}\int_{\mathbf{L}}\frac{\bar{F}_1(t)dt}{Y_0^+(t)(z-t)} \tag{4.13}$$

We now present formulae for the boundary values of the required complex functions which are useful for calculations. We will first indicate the limiting values of the auxiliary functions

$$\begin{aligned}
&\psi^\pm(x) = \pi w_j - \arg\{Z_0(x)\Pi_1(x)\} \pm i\psi_0(x), \quad x \in L_j \\
&\psi^\pm(x) = \psi_0(x), \quad x \in \mathbf{R}, \quad \psi_0(x) = \frac{Y^j(x)}{\pi}\int_{\mathbf{L}}\frac{\langle\eta_j(t)\rangle dt}{Y^j(t)(x-t)} \\
&Y^\pm(x) = Y^j(x)\times\begin{cases}\pm i, & x \in L_j \\ 1, & b_j < x < a_{j+1}\end{cases}, \quad |Z_0^\pm(x)| = \begin{cases}1, & x \in \mathbf{L}\cup\mathbf{T} \\ |\lambda|^{\pm 1/2}, & x \in \mathbf{S}\end{cases} \\
&Y^j(x) = (-1)^{J-j}|Y(x)|, \quad \arg Z_0^\pm(x) = \alpha\sum_{k=1}^{K}\ln\left|\frac{x-s_k}{x-t_k}\right|, \quad x \in \mathbf{L}\cup\mathbf{R} \\
&F_c^\pm(F_1^\pm + F_2^\pm) = \begin{cases} if^\pm + F_c^\pm(\hat{F}_1 + \hat{F}_2), & x \in \mathbf{L} \\ \pm \tfrac{1}{2}G^\pm f_R + F_c^\pm(F_1 + F_2), & x \in \mathbf{R}\end{cases} \\
&F^\pm(x) = F_c^\pm(x)\{P_r \pm Q_s Y_1 Y^j + \hat{F}_1 + \hat{F}_2\}(x) + if^\pm(x), \quad x \in \mathbf{L} \\
&F_c^\pm(x) = (-1)^{m_j^+}|\Pi(x)|\exp\{\mp\psi_0(x)\}\times\begin{cases}1, & x \in \mathbf{L}-\mathbf{L}^{**} \\ \mp 1, & x \in \mathbf{L}^{**}\end{cases} \\
&F^\pm(x) = F_c^\pm(x)\{P_r + iQ_s Y_1 Y^j + F_1 + F_2\}(x) \pm \frac{1}{2}G^\pm f_R(x), \quad x \in \mathbf{R} \\
&F_c^\pm(x) = (-1)^{K_+ + J_+}|\Pi(x)|\exp\{i\psi^\circ(x)\}\times\begin{cases}1, & x \in \mathbf{T} \\ \mp i, & x \in \mathbf{S}\end{cases} \\
&\psi^\circ(x) = \psi_0(x) + \arg\{Z_0(x)\Pi_1(x)\}, \quad G^+ = 1, \quad x \in \mathbf{R} \\
&G^- = 1, \quad x \in \mathbf{T}; \quad G^- = \lambda^{-1}, \quad x \in \mathbf{S}
\end{aligned} \tag{4.14}$$

Here, the integrals are understood in the sense of a principal value, if they do not exist in the Riemann sense. It is obvious from (4.14) that the functions $F^{\pm}(x)$ satisfy all the boundary conditions (4.2)–(4.4).

The symmetric stresses in the intervals of contact and the corresponding discontinuities in the derivatives of the displacements in the domains of incomplete contact and opening are expressed in terms of the boundary values of the function $F(z)$ as

$$\begin{aligned}
&\sigma = \varkappa(\lambda-1)\hat{F}^- - \beta_0\tau + \varkappa\hat{f}_S, \quad \tau = (\lambda-1)\tilde{F}^- + \tilde{f}_S \quad (\mathbf{S})\\
&\sigma = \varkappa(\hat{F}^+ - \hat{F}^-) - \beta_0\langle\tau_0\rangle, \quad [U] = \tilde{H}\{(\zeta+\varkappa)\hat{F}^+ + (\zeta-\varkappa)\hat{F}^-\} \quad (\mathbf{L})\\
&[V] = H_2\{2\varkappa\tilde{F}^+ + (\beta-\varkappa)f_1\}, \quad [U] = 2\zeta\tilde{H}\hat{F}^+ + \tilde{H}(1-d^{-1})f_2 + \beta_0[V] \quad (\mathbf{T})
\end{aligned} \tag{4.15}$$

The behaviour of the function $F(z)$ close to the singular points a is determined by the following asymptotic forms, where the terms $O(\ln(z-a))$ and $O(1)$ which ensure that the boundary conditions are satisfied locally, have been omitted:

$$\begin{aligned}
&F(z) \sim \{\Pi_0'\Pi_1\Pi_2\Pi_3\}(a)\exp\{i\psi(a)\}\{P_r + iQ_sY_0 + F_1 + F_2\}(a)\frac{(z-s_k)^{-1/2+i\alpha}}{(z-t_k)^{1/2+i\alpha}}\\
&z \to a = s_k,\ t_k \in \mathbf{S/T} \cup \mathbf{T/S}\\
&F^{\pm}(z) \sim i(-1)^{l_j^{\pm}}|\lambda|^{\pm\varepsilon_0}\{|\Pi|Y_0'(F_{21}+Q_s)\}(a)(z-a)^{-1/2}, \quad z \to a = b_n'\\
&l_j^{\pm} = m_j^{+},\quad b_n' \equiv b_j \in \mathbf{L/T};\quad l_j^{+} = m_j^{+}+1,\quad l_j^{-} = m_j^{+},\quad b_n' \equiv b_j \in \mathbf{L/S}\\
&F^{\pm}(z) \sim (-1)^{m_j^{+}+1}P(a)|\Pi_0\Pi_1\Pi_2\Pi_3'|(a)(z-a)^{-1/2}, \quad a = a_n' \in \mathbf{T/L}\\
&F^{\pm}(z) \sim (-1)^{m_j^{+}}|\lambda|^{\pm 1/2}P(a)\frac{|\Pi_0'\Pi_1\Pi_2\Pi_3|(a)}{\sqrt{z-a}} \times \begin{cases} i, & a = s_k \in \mathbf{L/S},\quad a \neq b_n' \\ \pm 1, & a = t_k \in \mathbf{S/L}\end{cases}\\
&P(a) = \{P_r + \hat{F}_1 + F_{22}\}(a),\quad j: a \equiv a_j, b_j;\quad a_j = t_k,\quad b_j = s_k\\
&F^{\pm}(z) \sim (-1)^{m_j^{+}}P(a)\frac{|\Pi_0\Pi_1\Pi_2'\Pi_3|(a)}{\sqrt{z-a}} \times \begin{cases} -1, & a = a_m'' \\ \mp i, & a = b_m''\end{cases}
\end{aligned} \tag{4.16}$$

where a prime on a product denotes that a pair (or, according to the sense, one) of the factors, which tends to zero or to infinity as the point investigated is approached, drops out from it. In the derivation, general methods for investigating the behaviour of Cauchy-type integrals with different singularities in the density have been used [11] and, in particular, the product of a logarithmic singularity and a power singularity as, for example, in the case of the function $\psi^{\pm}(z)$:

$$\psi^{\pm} = \pi w_l - \arg\Pi_1(a) - \alpha\ln\prod_{k=1}^{K}{}'\left|\frac{a-s_k}{a-t_k}\right| - \alpha\ln\frac{z-s_m}{z-t_m} \mp i\alpha\pi + O((z-a)^{1/2})$$

$$z \to a = t_m,\ s_m \equiv a_l,\ b_l \in S_m/L_l,\ L_l/S_m$$

The relation between the flow of energy into the cut tip and the stress intensity factors has been presented previously in [2], where it was pointed out that, unlike the case of a homogeneous plane, these quantities remain bounded and, generally speaking, are non-zero when $c_0 \to c_R$. The conditions for smooth separation at the nodal points T/L and L/T are ensured by the equalities which eliminate the singularity at these points shown in (4.16),

$$\begin{aligned}
&\{P_r + \hat{F}_1 + F_{22}\}(a) = 0, \quad a = a_l'',\ b_l'',\ a_n', \quad a_n' \in \mathbf{T/L'}\\
&\{F_{21} + Q_s\}(b_n') = 0, \quad b_n' \in \mathbf{L'/T}
\end{aligned} \tag{4.17}$$

from which these points of separation are determined. The liquidation of these singularities follows from the conventional constraints in the form of the inequalities: $[v] \geqslant 0$, in the domains of separation and $\sigma \leq 0$, in the domains of slip contact, although more quaint conditions are also possible when the boundaries of the slip zones are specified, the separation is not smooth and negative contact pressures are permitted. We now explain that Eqs (4.17) only ensure the correct combination of signs of the asymptotic forms of σ and $\{v\}$ to the left and right of the point of separation and an *a posteriori* global check of the above-mentioned inequalities is necessary when formulae (4.15) are used in order to discover the certainty of the correctness of the choice of numbers and the arrangement of the slip and open intervals in an actual problem. The uniqueness of the solutions of problems with alternative contact conditions has been proved in [12].

5. THE STEADY MOTION OF A CUT

As an illustration, we will consider the motion of a semi-infinite crack-cut along the boundary of separation under the action of a constant, oblique moving force $\pm(\Gamma_0, \Sigma_0)$, $\Sigma_0 > 0$ imposed at the points $x' = 0$, $y' = \pm 0$ of the stationary system of coordinates ($z' = x' + iy'$). The crack is open in the interval (a_1, b_1), conditions of frictionless contact are satisfied in the arc $(-\infty, a_1]$ and in the zone around the vertex of the cut tip $[b_1, s_1)$ and the anisotropic and different half planes $y' > 0$ and $y' < 0$ are bonded along the arc $[s_1, \infty)$. The positions of the points $a_1 < 0$, $b_1 > 0$ are determined during the course of the solution. For brevity, we will symmetrize the problem by means of the conformal mapping $z = (b + Az')(1 - Az')^{-1}$ with correspondence of the points $z' = \infty, s_1, a_1, b_1 \Leftrightarrow z = -1, 1, -a, a$. The open, slip and full contact conditions are now respectively satisfied in the intervals

$$(-a, a) = \mathbf{T}, \quad (-1, -a] \cup [a, 1) = L_1 \cup L_2 = \mathbf{L}, \quad (-\infty, -1] \cup [1, \infty) = \mathbf{S}$$

and the forces are applied at the points $x = b$, $y = \pm 0$; $-a < b < a$. We will seek the solution in the same class of functions as in (4.12) with the exception of the behaviour in the neighbourhood of the point $z = -1 \Leftrightarrow z' = \infty$ where we require

$$F(z) = O\{(z+1)^{3/2}\} + O\{(z+1)^2\} \tag{5.1}$$

This equality implies, in particular, that there is no pole at infinity in the function F in the z' plane. The real constants a and b are determined during the course of the solution and the quantities a_1, b_1 and A are then established using the formulae

$$a_1 = \frac{2s_1(a+b)}{(1-a)(b-1)}, \quad b_1 = \frac{2s_1(a-b)}{(1+a)(1-b)}, \quad A = \frac{1-b}{2s_1}$$

The procedure for removing the root singularities at the points $z = \pm a$ enables one to determine the two free constants P_0 and Q_0 in the solution. The integral, which determines the function $F_2(z)$ in (4.13), can be calculated using the theory of residues. We transform the integral (4.8) for $\psi(z)$ so that the oscillating function $Z_0(z)$ is completely annihilated in this case and, when account is taken of relations (4.13) and (5.1), solution (4.12) can then be written in the form

$$F = |\lambda|^{\pm 1/2} \frac{(z+1)^{3/2} \exp\{i\psi(z)\}}{(z+i\gamma)(b-z)} \left\{ \hat{F}_0 \frac{\sqrt{z-a}}{b-a} + i\tilde{F}_0 \sqrt{\frac{(1-b)(z+a)(z+1)}{(1+b)(a^2-b^2)(z-1)}} \right\} \tag{5.2}$$

$$F_0 = \pm \frac{(\Sigma_* + i\varkappa\Gamma_0)(b+i\gamma)(a-b)^{1/2}}{2\pi\varkappa(b+1)^{3/2}\exp\{i\psi_*\}}, \quad \Sigma_* = \Sigma_0 + \beta_0\Gamma_0, \quad \psi_* = \psi_0(b)$$

$$\psi = Y(z)\left\{ \int_L \frac{w_j + \pi^{-1}\operatorname{arctg}(\gamma/t)}{iY^+(t)(t-z)} dt + \int_{-a}^{a} \frac{\alpha}{Y^+(t)(t-z)} dt \right\}$$

$$Y(z) = \sqrt{(z^2-1)(z^2-a^2)}, \quad j = 1, 2, \quad w_1 = m_1^+ + 1/2, \quad w_2 = m_2^+$$

The signs $\pm$ in the second equality refer to the upper and lower half-planes, and, to calculate the functions at the interface, it is useful to know that

$$\psi^{\pm}(x)=\pi w_j+\operatorname{arctg}(\gamma/t)\pm i\psi_0(x)\quad (\mathbf{L}),\quad \psi^{\pm}(x)=\psi_0(x)\pm i\alpha\pi\quad (\mathbf{T})$$

$$\psi^{\pm}(x)=\psi_0(x)\quad (\mathbf{S})$$

$$\psi_0(x)=Y^j(x)\int\limits_{\mathbf{L}}\frac{w_j+\pi^{-1}\operatorname{arctg}(\gamma/t)}{Y^j(t)(x-t)}dt+\int\limits_{-a}^{a}\frac{\alpha Y^j(x)}{Y(t)(t-x)}dt$$

The expression for the functions $Y^j(x)$ is presented in (4.14). The condition that there is no pole at the point $z = c = -i\gamma$ gives two real equations for determining the constants a and b in terms of the quantity γ

$$a=\gamma^2,\quad \sqrt{\frac{(1+b)(\gamma^2+b)(\gamma^2+1)}{2(1-b)(\gamma^2-b)}}=\frac{\tilde{F}_0}{\hat{F}_0}=\frac{A_0\Sigma_*+\varkappa\Gamma_0}{\Sigma_*-\varkappa A_0\Gamma_0} \tag{5.3}$$

$$A_0=\frac{\gamma\cos\psi_*-b\sin\psi_*}{\gamma\sin\psi_*+b\cos\psi_*}$$

The solution of the equation in (5.3) is non-unique if the non-intersection condition is not taken into consideration. The principle behind the choice of the proper solution is as follows: we dwell on the solution which ensures the greatest permissible length of the slip zones around the the cut tip: $\max(s_1-b_1)$. It has been shown using a similar example [13] that the other solutions lead to a violation of the physical condition $[v]\geqslant 0$ in the interval of opening $(0, s_1)$.

In the unique auxiliary equation, which is analogous to (4.10), with the substitution $a = \gamma^2$, integration with respect to L_1 reduces to integration with respect to L_2 and, then, both integrals (over the intervals L_2 and $(-a, a)$) are reduced, by means of a change of variable, to an integral of a function with a weak singularity

$$\int\limits_0^{\pi/2}\left\{\frac{1}{q}\left(\operatorname{arctg}\left(\frac{\gamma}{q}\right)+\frac{\pi}{2}n^\circ\right)+\frac{2\alpha\pi}{\sqrt{1-\gamma^4\sin^2\phi}}\right\}d\phi=0 \tag{5.4}$$

$$q=\sqrt{1-(1-\gamma^4)\sin^2\phi}$$

The result can also be represented in terms of complete elliptic integrals.

The value γ and the difference $n^\circ = w_2 - w_1 = m_2^+ - m_1^+ - 1/2$ are uniquely defined from (5.4). One of the integers, for example, w_2, can be assigned arbitrarily and, then, $w_1 = w_2 - n^\circ$. It can be shown that, when w_2 is varied by an integer w, the integral determining the increment in the function ψ is taken and it will be equal to πw, and this can only lead to a change in the sign of the canonical solution.

The angular distribution of the stress intensity factors can be calculated, starting out from the asymptotic forms when $z\to 1$

$$F(z)=\pm\frac{4i\tilde{F}_0\,|\lambda|^{\pm 1/2}(z-1)^{-1/2}}{\sqrt{(1-b^2)(a^2-b^2)}}+O(1)$$

When $\lambda = -1$, $\alpha = 0$ (in particular, the same materials), the slip zones are lost: $a = 1$, $b_1 = s_1$, $a_1 = -\infty$. As the parameter α increases, the length of the opening zone decreases while, however, remaining finite on attaining a velocity c_R since, as $c\to c_R$, we have

$$|H_{1,2}|\to\infty,\quad \varkappa\to\varkappa_R\neq 0,\quad \infty,\quad \beta,\zeta\to 0,\quad \beta/\zeta\to 1,\quad \alpha\to\infty$$

This agrees with a qualitative analysis of the approximate solution of a similar problem for different isotropic materials where only a single slip interval close to the tip was taken into account [13].

In conclusion, we note that the solutions obtained, generally speaking, extend to a wider range of subsonic velocities. For this, it is necessary that the ellipticity of the initial system of equations should be preserved and it is necessary that an analysis is made of the existence and the relative distribution of the zeros of the characteristic functions, that is, the velocities of the waves localized around the interface and corresponding to the different contact conditions of the anisotropic half-planes. It follows from physical considerations that an increase in the degree of of constraint on the interface boundary leads to an increase in the velocities of the boundary waves, and this means that these velocities lie above the value c_R.

I wish to thank K. Yu. Osipenko for technical help in preparing the manuscript for publication.

This research was supported by the International Association for Promoting Cooperation with Scientists from the Independent States of the Former Soviet Union (INTAS-96-2306).

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Translated by E.L.S.